

PRINCIPAL CURVATURES AND PARALLEL SURFACES OF WAVE FRONTS

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ABSTRACT. We give criteria for which a principal curvature becomes a bounded C^∞ -function at non-degenerate singular points of wave fronts by using geometric invariants. As applications, we study singularities of parallel surfaces and extended distance squared functions of wave fronts. Moreover, we relate these singularities to some geometric invariants of fronts.

1. INTRODUCTION

In this paper, we study behavior of principal curvatures of wave fronts in the Euclidean 3-space \mathbf{R}^3 with non-degenerate singular points which contain cuspidal edges and swallowtails, and their applications.

Principal curvatures for regular surfaces play important roles to study the (extrinsic) differential geometry of surfaces and related topics. For instance, types of singularities of parallel surfaces and focal surfaces of regular surfaces are closely related to critical points of principal curvatures, such points are called ridge points. In fact, Porteous [20] showed relations between cuspidal edges on a focal surface and ridge points on an initial surface by using singularity theory techniques (cf. [5, 6]). Thus we expect that principal curvatures may play an important role to study wave fronts from the differential geometric viewpoint. In [17], Murata and Umehara showed that at least one principal curvature is unbounded near a singular point. However, another principal curvature may be bounded C^∞ -function. In this paper, we give an explicit criterion for which a principal curvature becomes a bounded C^∞ -function near non-degenerate singular points of wave fronts in terms of geometric invariants (Theorem 3.1). Using this relation, we define analogies of ridge points for wave front (Definition 3.3). This kind of criteria for the case of cuspidal edges is given in [28, Proposition 2.2].

As an application, we consider singularities of parallel surfaces on wave fronts. We studied parallel surfaces of cuspidal edges and gave criteria for swallowtails appearing on parallel surfaces of cuspidal edges in terms of geometric properties of cuspidal edges in [28]. However, we have not given criteria for other singularities which appear on parallel surfaces of cuspidal edges or wave fronts, in their differential geometric contexts. Thus we show relations between types of singularities of parallel surfaces on wave fronts and geometric properties of initial fronts (Theorem 4.2). Moreover, we give geometric relations between initial cuspidal edges and cuspidal edges which appear as singularities of parallel surfaces (Propositions 4.4, 4.5 and 4.6). In addition, we consider constant principal curvature (CPC) lines near singular points of wave fronts. Using parallel surfaces, we define special points (landmark in the sense of [21]) on cuspidal edge as cusps of CPC lines, which seems not to have appeared in the literature (Subsection 4.3).

Finally, we treat the extended distance squared function on wave fronts. For the case of generic regular surfaces, singularities of extended distance squared functions correspond to types of singularities of parallel surfaces (cf. [5, Theorem 3.4]). However, for wave fronts, the same statement does not hold, in fact, different kinds of singularities (D -type) will appear (Theorem 5.3).

All maps and functions considered here are of class C^∞ unless otherwise stated.

2. PRELIMINARIES

2.1. Wave fronts. We recall some properties of wave fronts. For details, see [1, 4, 9, 16, 25].

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A map $f : V \rightarrow \mathbf{R}^3$ is called a *wave front* (or a *front*) if there exists a unit normal vector ν to f such that the pair $L_f = (f, \nu) : V \rightarrow \mathbf{R}^3 \times S^2$ gives an immersion, where $V \subset (\mathbf{R}^2; u, v)$ is a domain and S^2 denotes the unit sphere in \mathbf{R}^3 (cf. [1, 12, 25]). A map $f : V \rightarrow \mathbf{R}^3$ is called a *frontal* if just a unit normal vector ν to f exists. A point p is said to be a *singular point* of f if f is not an immersion at p . We denote by $S(f)$ the set of singular points of f .

For a frontal f , the function $\lambda : V \rightarrow \mathbf{R}$ as

$$\lambda(u, v) = \det(f_u, f_v, \nu)(u, v) \quad (f_u = \partial f / \partial u, f_v = \partial f / \partial v)$$

is called the *signed area density function* (cf. [12, 25]). By the definition of λ , $S(f) = \lambda^{-1}(0)$ holds. We call $p \in S(f)$ a *non-degenerate* if $d\lambda(p) \neq 0$. Let p be non-degenerate. Then there exists a regular curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow V$ with $\gamma(0) = p$ such that γ locally parametrizes $S(f)$. Moreover, there exists a vector field η such that $df(\eta) = \mathbf{0}$ along γ . We call γ and η the *singular curve* and the *null vector field*, respectively. A non-degenerate singular point p is said to be of the *first kind* if $\eta(0)$ is transverse to $\gamma'(0)$, that is, $\det(\gamma', \eta)(0) \neq 0$. Otherwise, it is said to be of the *second kind* ([16]). Moreover, we call a non-degenerate singular point of the second kind *admissible* if the singular curve consists of points of the first kind except at p . Otherwise, we call p the *non-admissible*.

Definition 2.1. Let $f : (V, p) \rightarrow (\mathbf{R}^3, f(p))$ be a map-germ around p . Then f at p is a *cuspidal edge* if the map-germ f is \mathcal{A} -equivalent to the map-germ $(u, v) \mapsto (u, v^2, v^3)$ at $\mathbf{0}$, f at p is a *swallowtail* if the map-germ f is \mathcal{A} -equivalent to the map-germ $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at $\mathbf{0}$, f at p is a *cuspidal butterfly* if the map-germ f is \mathcal{A} -equivalent to the map-germ $(u, v) \mapsto (u, 4v^5 + uv^2, 5v^4 + 2uv)$ at $\mathbf{0}$, f at p is a *cuspidal lips* if the map-germ f is \mathcal{A} -equivalent to the map-germ $(u, v) \mapsto (u, 3v^4 + 2u^2v^2, v^3 + u^2v)$ at $\mathbf{0}$ and f at p is a *cuspidal beaks* if the map-germ f is \mathcal{A} -equivalent to the map-germ $(u, v) \mapsto (u, 3v^4 - 2u^2v^2, v^3 - u^2v)$ at $\mathbf{0}$, where two map-germs $f, g : (\mathbf{R}^2, \mathbf{0}) \rightarrow (\mathbf{R}^3, \mathbf{0})$ are \mathcal{A} -equivalent if there exist diffeomorphism-germs $\theta : (\mathbf{R}^2, \mathbf{0}) \rightarrow (\mathbf{R}^2, \mathbf{0})$ on the source and $\Theta : (\mathbf{R}^3, \mathbf{0}) \rightarrow (\mathbf{R}^3, \mathbf{0})$ on the target such that $\Theta \circ f = g \circ \theta$ holds.

We note that generic singularities of fronts are cuspidal edges and swallowtails and generic singularities of one-parameter bifurcation of fronts are cuspidal lips/beaks, cuspidal butterflies and D_4^\pm singularities in addition to above two (see [1, 9]). Here D_4^\pm singularities are map-germs \mathcal{A} -equivalent to $(u, v) \mapsto (uv, u^2 \pm 3v^2, u^2v \pm v^3)$ at $\mathbf{0}$.

Fact 2.2 ([10, 11, 12, 24]). *Let $f : (V, p) \rightarrow \mathbf{R}^3$ be a front germ, ν a unit normal to f and p a corank one singular point.*

- (1) *Suppose that p is a non-degenerate singular point.*
 - *f at p is \mathcal{A} -equivalent to a cuspidal edge if and only if $\eta\lambda(p) \neq 0$.*
 - *f at p is \mathcal{A} -equivalent to a swallowtail if and only if $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$.*
 - *f at p is \mathcal{A} -equivalent to a cuspidal butterfly if and only if $\eta\lambda(p) = \eta\eta\lambda(p) = 0$ and $\eta\eta\eta\lambda(p) \neq 0$.*
- (2) *Suppose that p is a corank one degenerate singular point.*
 - *f at p is \mathcal{A} -equivalent to a cuspidal lips if and only if $\det \mathcal{H}_\lambda(p) > 0$.*
 - *f at p is \mathcal{A} -equivalent to a cuspidal beaks if and only if $\eta\eta\lambda(p) \neq 0$ and $\det \mathcal{H}_\lambda(p) < 0$.*

Here λ is the signed area density function, η the null vector field and \mathcal{H}_λ the Hessian matrix of λ .

Remark 2.3. Cuspidal edges are non-degenerate singular points of the first kind. On the other hand, swallowtails and cuspidal butterflies are of the admissible second kind (cf. [16]). Thus generic singularities of fronts are admissible.

We note that there is a criterion for a *cuspidal cross cap* which appears on a frontal surface defined as a map-germ \mathcal{A} -equivalent to $(u, v) \mapsto (u, v^2, uv^3)$ at $\mathbf{0}$ ([4, Theorem 1.4]).

We recall behavior of curvatures of fronts near non-degenerate singular points p . Let $f : V \rightarrow \mathbf{R}^3$ be a front and ν a unit normal vector. Let K and H denote the Gaussian and the mean curvature of a front f . It is known that H is unbounded near p ([25, Corollary 3.5]). On the other hand, for the Gaussian curvature K , it is known that K is bounded near p if and only if the second fundamental form vanishes along the singular curve γ ([25, Theorem 3.1]). For K and H , the notions called the *rationally bounded* and the *rationally continuous* are introduced in [16].

Next we recall behavior of principal curvature maps of a front f at singular points. Let us assume that there are no umbilic points on V . Then there exists a local coordinate system $(U; u, v)$ centered at p such that f_u and ν_u (resp. f_v and ν_v) are linearly dependent on U . In particular, the pair $\{f_u, \nu_u\}$ (resp. $\{f_v, \nu_v\}$) does not vanish at the same time ([17, Lemma 1.3]). Such a coordinate system is called a *principal curvature line coordinate* introduced in [17]. For this local coordinate system $(U; u, v)$, we define the maps $\Lambda_i : U \rightarrow P^1(\mathbf{R})$ ($i = 1, 2$) which are called the *principal curvature maps* ([17]) as the proportional ratio of the real projective line $P^1(\mathbf{R})$ by

$$\Lambda_1 = [-\nu_u : f_u], \quad \Lambda_2 = [-\nu_v : f_v].$$

Fact 2.4 ([17, Lemma 1.7]). *Let $f : V \rightarrow \mathbf{R}^3$ be a front and Λ_1, Λ_2 be the principal curvature maps. Then $p \in V$ is a singular point if and only if either $\Lambda_1(p) = [1 : 0]$ or $\Lambda_2(p) = [1 : 0]$ holds.*

By Fact 2.4, one principal curvature function of a wave front is bounded and the other is unbounded near a singular point.

2.2. Invariants of a cuspidal edge. Let $f : V \rightarrow \mathbf{R}^3$ be a frontal, $p \in V$ a non-degenerate singular point and ν a unit normal vector. Then we can take the following local coordinate system around p .

Definition 2.5 ([12, 16, 25]). A local coordinate system $(U; u, v)$ centered at a singular point of the first kind (resp. of the second kind) p is called *adapted* if it is compatible with the orientation of V and satisfies the following conditions:

- (1) the u -axis is the singular curve,
- (2) $\eta = \partial_v$ (resp. $\eta = \partial_u + \varepsilon(u)\partial_v$ with $\varepsilon(0) = 0$) gives the null vector field on the u -axis,
- (3) there are no singular points other than the u -axis.

Let p be a cuspidal edge and $(U; u, v)$ an adapted coordinate system centered at p . Since $df(\eta) = f_v = \mathbf{0}$ along the u -axis, there exists a map $\varphi : U \rightarrow \mathbf{R}^3 \setminus \{\mathbf{0}\}$ such that $f_v = v\varphi$. We note that $f_{vv} = \varphi$ holds along the u -axis. Since $\eta\lambda = \det(f_u, \varphi, \nu) \neq 0$ on the u -axis by Fact 2.2, the pair $\{f_u, \varphi, \nu\}$ gives a frame (cf. [16, 28]).

Lemma 2.6 ([28, Lemma 2.1]). *It holds that*

$$\nu_u = \frac{\tilde{F}\tilde{M} - \tilde{G}\tilde{L}}{\tilde{E}\tilde{G} - \tilde{F}^2}f_u + \frac{\tilde{F}\tilde{L} - \tilde{E}\tilde{M}}{\tilde{E}\tilde{G} - \tilde{F}^2}\varphi, \quad \nu_v = \frac{\tilde{F}\tilde{N} - v\tilde{G}\tilde{M}}{\tilde{E}\tilde{G} - \tilde{F}^2}f_u + \frac{v\tilde{F}\tilde{M} - \tilde{E}\tilde{N}}{\tilde{E}\tilde{G} - \tilde{F}^2}\varphi,$$

where $\tilde{E} = \|f_u\|^2$, $\tilde{F} = \langle f_u, \varphi \rangle$, $\tilde{G} = \|\varphi\|^2$, $\tilde{L} = -\langle f_u, \nu_u \rangle$, $\tilde{M} = -\langle \varphi, \nu_u \rangle$ and $\tilde{N} = -\langle \varphi, \nu_v \rangle$.

For cuspidal edges, several geometric invariants are studied (for example, see [15, 16, 18, 25, 26, 27]). By using an adapted coordinate system $(U; u, v)$ and the frame $\{f_u, \varphi, \nu\}$, we set the following invariants along the u -axis:

$$\begin{aligned} \kappa_s(u) &= \text{sgn}(\lambda_v) \frac{\det(f_u, f_{uu}, \nu)}{\|f_u\|^3}(u, 0), \quad \kappa_\nu(u) = \frac{\langle f_{uu}, \nu \rangle}{\|f_u\|^2}(u, 0), \quad \kappa_c(u) = \frac{\|f_u\|^{3/2} \det(f_u, \varphi, f_{vvv})}{\|f_u \times \varphi\|^{5/2}}(u, 0), \\ \kappa_t(u) &= \frac{\det(f_u, \varphi, f_{uvv})}{\|f_u \times \varphi\|^2}(u, 0) - \frac{\det(f_u, \varphi, f_{uu})\langle f_u, \varphi \rangle}{\|f_u\|^2 \|f_u \times \varphi\|^2}(u, 0). \end{aligned}$$

$\kappa_s, \kappa_\nu, \kappa_c$ and κ_t are called the *singular curvature*, the *limiting normal curvature*, the *cuspidal curvature* and the *cusp-directional torsion*, respectively. See [7, 15, 16, 25, 27] for details of their geometric meanings. We note that these invariants can be defined on frontals with singular points of the first kind.

Lemma 2.7. *Under the above settings, κ_ν, κ_c and κ_t can be expressed as*

$$(2.1) \quad \kappa_\nu(u) = \frac{\tilde{L}}{\tilde{E}}(u, 0), \quad \kappa_c(u) = \pm \frac{2\tilde{E}^{3/4}\tilde{N}}{(\tilde{E}\tilde{G} - \tilde{F}^2)^{3/4}}(u, 0), \quad \kappa_t(u) = \pm \frac{\tilde{E}\tilde{M} - \tilde{F}\tilde{L}}{\tilde{E}\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(u, 0)$$

along the u -axis, where \pm depends on the orientation of the frame $\{f_u, \varphi, \nu\}$.

Proof. One can check that κ_ν can be expressed as above by definitions of functions. We show κ_c and κ_t can be written as above formulas. Since ν is perpendicular to both f_u and φ , ν can be written as $\nu = \pm(f_u \times \varphi)/\|f_u \times \varphi\|$.

First, we show that κ_c can be written as above. We note that $f_{vvv} = 2\varphi_v$ holds on the u -axis. Since $\tilde{N} = -\langle \varphi, \nu_v \rangle = \langle \varphi_v, \nu \rangle$, κ_c on the u -axis is expressed as

$$\kappa_c(u) = \frac{2\tilde{E}^{3/4} \det(f_u, \varphi, \varphi_v)}{\|f_u \times \varphi\|^{5/2}}(u, 0) = \pm \frac{2\tilde{E}^{3/4} \langle \nu, \varphi_v \rangle}{\|f_u \times \varphi\|^{3/2}}(u, 0) = \pm \frac{2\tilde{E}^{3/4} \tilde{N}}{(\tilde{E}\tilde{G} - \tilde{F}^2)^{3/4}}(u, 0)$$

on the u -axis.

Next, we consider κ_t . Since $f_{uvv} = \varphi_u$ and $\langle \varphi_u, \nu \rangle = -\langle \varphi, \nu_u \rangle = \tilde{M}$ on the u -axis, we see that

$$\kappa_t(u) = \frac{\det(f_u, \varphi, \varphi_u)}{\tilde{E}\tilde{G} - \tilde{F}^2}(u, 0) - \frac{\det(f_u, \varphi, f_{uu})\tilde{F}}{\tilde{E}(\tilde{E}\tilde{G} - \tilde{F}^2)}(u, 0) = \pm \frac{\tilde{E}\tilde{M} - \tilde{F}\tilde{L}}{\tilde{E}\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(u, 0).$$

□

It is known that $\kappa_c(p)$ does not vanish if p is a cuspidal edge (cf. [16, Lemma 2.11]). In particular, \tilde{N} never vanishes on the u -axis by Lemma 2.7. Take an adapted coordinate system $(U; u, v)$ with $\eta\lambda(u, 0) > 0$. Then $\text{sgn}(\kappa_c) = \text{sgn}(\tilde{N})$ holds on the u -axis (see Lemma 2.7). If $\eta\lambda(u, 0) < 0$, $\text{sgn}(\kappa_c) = -\text{sgn}(\tilde{N})$ holds.

We define the following functions on $U \setminus \{v = 0\}$ as

$$(2.2) \quad \kappa_+ = \frac{2(\tilde{L}\tilde{N} - v\tilde{M}^2)}{\tilde{A} + \tilde{B}}, \quad \kappa_- = \frac{2(\tilde{L}\tilde{N} - v\tilde{M}^2)}{\tilde{A} - \tilde{B}},$$

where $\tilde{A} = \tilde{E}\tilde{N} - 2v\tilde{F}\tilde{M} + v\tilde{G}\tilde{L}$, $\tilde{B} = \sqrt{\tilde{A}^2 - 4v(\tilde{E}\tilde{G} - \tilde{F}^2)(\tilde{L}\tilde{N} - v\tilde{M}^2)}$. These functions are well-defined on $U \setminus \{v = 0\}$. We remark that κ_+ (resp. κ_-) becomes $-\kappa_-$ (resp. $-\kappa_+$) if we change ν to $-\nu$. Let K and H be the Gaussian and the mean curvature of f defined on $U \setminus \{v = 0\}$. Then $K = \kappa_+\kappa_-$ and $2H = \kappa_+ + \kappa_-$ hold. Thus we may treat κ_+ and κ_- as *principal curvatures* of f defined on $U \setminus \{v = 0\}$. Here K and H can be expressed as

$$K = \frac{\tilde{L}\tilde{N} - v\tilde{M}^2}{v(\tilde{E}\tilde{G} - \tilde{F}^2)}, \quad H = \frac{\tilde{E}\tilde{N} - 2v\tilde{F}\tilde{M} + v\tilde{G}\tilde{L}}{2v(\tilde{E}\tilde{G} - \tilde{F}^2)}$$

on the set of regular points. We note that $\kappa_{\pm} = H \mp \sqrt{H^2 - K}$ hold on the set of regular points.

2.3. Invariants of a singular point of the second kind. Let $f : V \rightarrow \mathbf{R}^3$ be a frontal, p a non-degenerate singular point of the second kind and ν a unit normal vector to f . We fix an adapted coordinate system $(U; u, v)$ in the following (see Definition 2.5). Taking a null vector field η , there exists a function $\varepsilon = \varepsilon(u)$ on the u -axis with $\varepsilon(0) = 0$ so that $\eta = \partial_u + \varepsilon(u)\partial_v$ (see [16]). (We note that $\varepsilon \equiv 0$ holds on the u -axis, namely, $\eta = \partial_u$ if p is of the non-admissible.) Thus it follows that $df(\eta) = f_u + \varepsilon(u)f_v = \mathbf{0}$ holds along the u -axis. On the other hand, since the u -axis gives the singular curve, there exists a C^∞ -function $\varphi : U \rightarrow \mathbf{R}^3 \setminus \{\mathbf{0}\}$ such that $df(\eta) = v\varphi$. Hence we have $f_u = v\varphi - \varepsilon f_v$. We remark that φ , f_v and ν are linearly independent since $d\lambda = \det(\varphi, f_v, \nu)dv \neq 0$ holds on the u -axis.

Lemma 2.8. *Under the adapted coordinate system $(U; u, v)$, ν_u and ν_v on U can be written as*

$$\nu_u = \frac{\hat{F}(v\hat{M} - \varepsilon\hat{N}) - \hat{G}\hat{L}}{\hat{E}\hat{G} - \hat{F}^2}\varphi + \frac{\hat{F}\hat{L} - \hat{E}(v\hat{M} - \varepsilon\hat{N})}{\hat{E}\hat{G} - \hat{F}^2}f_v, \quad \nu_v = \frac{\hat{F}\hat{N} - \hat{G}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2}\varphi + \frac{\hat{F}\hat{M} - \hat{E}\hat{N}}{\hat{E}\hat{G} - \hat{F}^2}f_v,$$

where $\hat{E} = \|\varphi\|^2$, $\hat{F} = \langle \varphi, f_v \rangle$, $\hat{G} = \|f_v\|^2$, $\hat{L} = -\langle \varphi, \nu_u \rangle$, $\hat{M} = -\langle \varphi, \nu_v \rangle$ and $\hat{N} = -\langle f_v, \nu_v \rangle$.

We now define two C^∞ -functions on $U \setminus \{v = 0\}$ by

$$(2.3) \quad \kappa_+ = \frac{2((\hat{L} + \varepsilon(u)\hat{M})\hat{N} - v\hat{M}^2)}{\hat{A} + \hat{B}}, \quad \kappa_- = \frac{2((\hat{L} + \varepsilon(u)\hat{M})\hat{N} - v\hat{M}^2)}{\hat{A} - \hat{B}},$$

where

$$\begin{aligned} \hat{A} &= \hat{G}(\hat{L} + \varepsilon(u)\hat{M}) - 2v\hat{F}\hat{M} + v\hat{E}\hat{N}, \\ \hat{B} &= \sqrt{\hat{A}^2 - 4v(\hat{E}\hat{G} - \hat{F}^2)((\hat{L} + \varepsilon(u)\hat{M})\hat{N} - v\hat{M}^2)}. \end{aligned}$$

Since the Gaussian curvature K and the mean curvature H of f satisfy $K = \kappa_+ \kappa_-$ and $2H = \kappa_+ + \kappa_-$, we may regard κ_{\pm} as *principal curvatures* of f on $U \setminus \{v = 0\}$, where K and H are written as

$$K = \frac{(\hat{L} + \varepsilon(u)\hat{M})\hat{N} - v\hat{M}^2}{v(\hat{E}\hat{G} - \hat{F}^2)}, \quad H = \frac{\hat{G}(\hat{L} + \varepsilon(u)\hat{M}) - 2v\hat{F}\hat{M} + v\hat{E}\hat{N}}{2v(\hat{E}\hat{G} - \hat{F}^2)}$$

on $U \setminus \{v = 0\}$. We remark that $\kappa_{\pm} = H \mp \sqrt{H^2 - K}$ hold on the set of regular points.

We put $\hat{H} = vH$. This is a C^∞ -function on U . It follows that

$$(2.4) \quad 2\hat{H} = \frac{\hat{G}(\hat{L} + \varepsilon(u)\hat{M})}{\hat{E}\hat{G} - \hat{F}^2}$$

holds along the u -axis (cf. [16]). We note that $\hat{L} + \varepsilon(u)\hat{M} = -\langle \varphi, \eta\nu \rangle$ holds. It is known that $2\hat{H}$ does not vanish on the u -axis if and only if f is a front ([16, Proposition 3.2]). We set

$$\mu_c(p) = 2\hat{H}(p) \left(= \frac{\hat{G}(p)\hat{L}(p)}{\|\varphi(p) \times f_v(p)\|^2} \right).$$

This is a geometric invariant called the *normalized cuspidal curvature* defined in [16]. By (2.4) and the definition of $\mu_c(p)$, we see that $\text{sgn}(\mu_c(p)) = \text{sgn}(\hat{L}(p))$ and $\hat{L}(p) \neq 0$ hold if f is a front.

Lemma 2.9. *Under the above conditions, the limiting normal curvature κ_ν can be written as $\kappa_\nu = \hat{N}/\hat{G}$ at p if p is of the admissible second kind.*

Proof. By [16, Proposition 1.9], $f_u = v\varphi - \varepsilon(u)f_v$, $f_{uu} = v\varphi_u - \varepsilon'(u)f_v - \varepsilon(u)f_{uv}$ and $f_{uv} = \varphi + v\varphi_v - \varepsilon(u)f_{vv}$, we get the conclusion. \square

3. PRINCIPAL CURVATURES, PRINCIPAL VECTORS AND RIDGE POINTS

3.1. Boundedness of a principal curvature. In this subsection, we consider boundedness of principal curvatures of fronts by using the above arguments.

Theorem 3.1. *Let $f : V \rightarrow \mathbf{R}^3$ be a front and p a non-degenerate singular point.*

- (1) *Let p be a cuspidal edge.*
 - *Suppose that $\eta\lambda(p) > 0$. If $\kappa_c(p) > 0$, then the principal curvature κ_+ is a bounded C^∞ -function at p . Moreover, $\kappa_+(p) = \kappa_\nu(p)$.*
 - *Suppose that $\eta\lambda(p) < 0$. If $\kappa_c(p) < 0$, then the principal curvature κ_+ is a bounded C^∞ -function at p . Moreover, $\kappa_+(p) = \kappa_\nu(p)$ (resp. $\kappa_-(p) = \kappa_\nu(p)$).*
- (2) *Let p be of the second kind. If $\mu_c(p) > 0$, then the principal curvature κ_+ is a bounded C^∞ -function at p . Moreover, $\kappa_+(p) = \kappa_\nu(p)$ if p is an admissible.*

Converses are also true. Moreover, if one of κ_{\pm} is bounded at p , then the another is unbounded.

Proof. We prove the first assertion. Let $f : V \rightarrow \mathbf{R}^3$ be a front and p a cuspidal edge. Take an adapted coordinate system $(U; u, v)$ centered at p . We show the case of $\eta\lambda(u, 0) > 0$. In this case, $\text{sgn}(\kappa_c) = \text{sgn}(\tilde{N})$ holds along the u -axis. For the case of $\eta\lambda(u, 0) < 0$, one can show similarly.

We now assume that $\kappa_c(p) > 0$. Then $\tilde{N}(p) > 0$ by (2.1). Since $\tilde{A} \pm \tilde{B} = \tilde{E}(\tilde{N} \pm |\tilde{N}|)$ and (2.2), we see that κ_+ is a bounded C^∞ -function on U and $\kappa_+ = \tilde{L}/\tilde{E} = \kappa_\nu$ holds at p . Conversely, we assume that the principal curvature κ_+ is a bounded C^∞ -function near p . In this case, it follows that $\tilde{N} = -\langle \varphi, \eta\nu \rangle$ is positive along the u -axis. This implies that κ_c is positive along the u -axis by (2.1). Unboundedness of κ_- near p follows from the fact that the mean curvature is unbounded near p .

Next, we prove the second assertion. Take an adapted coordinate system $(U; u, v)$ centered at a non-degenerate singular point of the second kind p . Suppose that $\mu_c(p) = 2\hat{H}(p) > 0$. It follows that $-\langle \varphi, \eta\nu \rangle > 0$ holds near p from (2.4). Since $\hat{A} = \hat{G}(-\langle \varphi, \eta\nu \rangle)$, $\hat{B} = |\hat{A}|$ and $-\langle \varphi, \eta\nu \rangle > 0$ along the u -axis, it follows that $\hat{A} + \hat{B} = 2\hat{G}(-\langle \varphi, \eta\nu \rangle) \neq 0$ and $A - B = 0$ hold on the u -axis. Hence by (2.3), we have $\kappa_+ = \hat{N}/\hat{G}$ along the u -axis, and κ_+ is a C^∞ -function. By Lemma 2.9, we see that $\kappa_+ = \kappa_\nu$ at p if p is an admissible. The converse and unboundedness can be shown by using similar arguments to (1). \square

3.2. Principal vectors and ridge points. By Theorem 3.1, one of κ_{\pm} of fronts can be defined as a bounded C^{∞} -function near non-degenerate singular points. This implies there is a principal vector with respect to such a principal curvature at the singular point. Hence we consider explicit representation of the principal vector under an adapted coordinate system.

Let $f : V \rightarrow \mathbf{R}^3$ be a front, p a singular point of the second kind and ν a unit normal vector to f . Then we take an adapted coordinate system $(U; u, v)$ around p . Assume that $\mu_c(p) > 0$, namely, κ_+ is a bounded C^{∞} -function near p in the following. We investigate the principal vector relative to κ_+ .

Let I and II denote the first and the second fundamental matrices given by

$$I = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_u, f_v \rangle & \langle f_v, f_v \rangle \end{pmatrix}, \quad II = \begin{pmatrix} -\langle f_u, \nu_u \rangle & -\langle f_u, \nu_v \rangle \\ -\langle f_v, \nu_u \rangle & -\langle f_v, \nu_v \rangle \end{pmatrix}.$$

The principal vector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ with respect to κ_+ is a never vanishing vector satisfying $(II - \kappa_+ I)\mathbf{v} = \mathbf{0}$. We can write this equation as

$$(3.1) \quad \begin{pmatrix} v\{\hat{L} - \kappa_+(v\hat{E} - \varepsilon\hat{F})\} & v(\hat{M} - \kappa_+\hat{F}) \\ v(\hat{M} - \kappa_+\hat{F}) - \varepsilon(\hat{N} - \kappa_+\hat{G}) & \hat{N} - \kappa_+\hat{G} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We note that \hat{L} does not vanish at p . Thus we can take the principal vector \mathbf{v} as

$$(3.2) \quad \mathbf{v} = (-\hat{M} + \kappa_+\hat{F}, \hat{L} - \kappa_+(v\hat{E} - \varepsilon\hat{F})),$$

by factoring out v from (3.1). For the case of cuspidal edges, the principal vector \mathbf{v} with respect to κ_+ is given as follows [28]:

$$(3.3) \quad \mathbf{v} = (\tilde{N} - v\kappa_+\tilde{G}, -\tilde{M} + \kappa_+\tilde{F}).$$

We can extend the notion of a line of curvature as follows. The singular curve γ is a *line of curvature* if the principal vector \mathbf{v} is tangent to γ .

Proposition 3.2. *Let $f : V \rightarrow \mathbf{R}^3$ be a front, p a non-degenerate singular point and γ the singular curve passing through p . Then the following assertions hold:*

- (1) *Suppose that p is a cuspidal edge. Then γ is a line of curvature of f if and only if κ_t vanishes identically along γ .*
- (2) *Suppose that p is of the second kind. Then γ can not be a line of curvature.*

Proof. First, we show assertion (1). Take an adapted coordinate system $(U; u, v)$ centered at a cuspidal edge p satisfying $\eta\lambda(u, 0) > 0$. Assume that κ_+ is bounded on U . Then the principal vector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ relative to κ_+ is given by (3.3). Since $\kappa_+ = \tilde{L}/\tilde{E}$ on the u -axis, \mathbf{v}_2 can be written as

$$\mathbf{v}_2 = -\tilde{M} + \kappa_+\tilde{F} = -\frac{\tilde{E}\tilde{M} - \tilde{F}\tilde{L}}{\tilde{E}} = -\kappa_t\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}$$

along the u -axis by Lemma 2.7. Thus \mathbf{v}_2 vanishes on the u -axis if and only if κ_t vanishes along the u -axis, and we get the conclusion.

Next, we show (2). Take an adapted coordinate system $(U; u, v)$ around p and assume that $\mu_c(p) > 0$ holds. In this case, κ_+ is bounded on U and the principal vector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ of κ_+ is given as (3.2). The second component \mathbf{v}_2 is written as

$$\mathbf{v}_2 = \hat{L} + \varepsilon\kappa_+\hat{F}$$

along the u -axis. Thus we have $\mathbf{v}_2 = \hat{L} \neq 0$ at p . This implies that the u -axis can not be the line of curvature. \square

Using the principal curvature κ_+ and the principal vector \mathbf{v} relative to κ_+ , we define ridge points for f . Ridge points play important role to study parallel surfaces.

Definition 3.3. Under the above settings, a point p is called a *ridge point* if $\mathbf{v}\kappa_+(p) = 0$ holds, where $\mathbf{v}\kappa_+$ denotes the directional derivative of κ_+ with respect to \mathbf{v} . Moreover, a point p is called a *k -th order ridge point* if $\mathbf{v}^{(m)}\kappa_+(p) = 0$ ($1 \leq m \leq k$) and $\mathbf{v}^{(k+1)}\kappa_+(p) \neq 0$ hold, where $\mathbf{v}^{(m)}\kappa_+$ means the m -th directional derivative of κ_+ with respect to \mathbf{v} .

Ridge points for regular surfaces were first studied deeply by Porteous [20]. He showed that ridge points correspond to A_3 singular points, that is, cuspidal edges of caustics. For more details on ridge points, see [3, 5, 6, 9, 20, 21].

4. PARALLEL SURFACES OF WAVE FRONTS

For the case of regular surfaces, principal curvatures relate singularities of parallel surfaces. In this section, we consider singularities of parallel surfaces of fronts and give criteria in terms of principal curvatures and other geometric properties. Swallowtails on parallel surfaces of cuspidal edges are studied in [28]. Here we give criteria for other singularities on parallel surfaces of fronts.

4.1. Singularities of parallel surfaces of wave fronts. In this subsection, we shall deal with fronts which have singular points of the second kind (swallowtails, for example). Needless to say, the following arguments can be applied to the case of cuspidal edges.

Let $f : V \rightarrow \mathbf{R}^3$ be a front, ν a unit normal to f and $p \in V$ a non-degenerate singular point of the second kind. Then the *parallel surface* f^t of f is defined by $f^t = f + t\nu$, where $t \in \mathbf{R} \setminus \{0\}$ is constant. We note that f^t is also a front since ν is a unit normal to f^t .

Lemma 4.1. *Let $f : V \rightarrow \mathbf{R}^3$ be a front, ν its unit normal vector and p a non-degenerate singular point of f . Suppose that κ_+ is a bounded C^∞ -function near p and $\kappa_+(p) \neq 0$. Then p is a singular point of f^t if and only if $t = 1/\kappa_+(p)$. Moreover, p is non-degenerate if and only if p is not a critical point of κ_+ .*

Proof. We show the case that p is of the second kind. Let $(U; u, v)$ be an adapted coordinate system centered at p with the null vector field $\eta = \partial_u + \varepsilon(u)\partial_v$. Then the signed area density function $\lambda^t = \det(f_u^t, f_v^t, \nu)$ of f^t can be written as

$$\lambda^t = \det(f_u^t, f_v^t, \nu) = (1 - t\kappa_+)(\lambda - t\lambda\kappa_-)$$

by Lemma 2.8, where $\lambda = \det(f_u, f_v, \nu)$. Since $\lambda\kappa_-$ does not vanish at p , p is a singular point of f^t if and only if $t = 1/\kappa_+(p)$ holds. Thus we may treat $\hat{\lambda}^t = \kappa_+(u, v) - \kappa_+(p)$ as the signed area density function of f^t . Non-degeneracy follows $d\hat{\lambda}^t = (\kappa_+)_u du + (\kappa_+)_v dv$. \square

Theorem 4.2. *Let $f : V \rightarrow \mathbf{R}^3$ be a front and p be a non-degenerate singular point. Suppose that the principal curvature κ_+ is a bounded C^∞ -function on V and $\kappa_+(p) \neq 0$. Then for the parallel surface f^t with $t = 1/\kappa_+(p)$, the following conditions hold.*

- (1) *Assume $d\kappa_+(p) \neq 0$. Then the following hold:*
 - *The map-germ f^t at p is \mathcal{A} -equivalent to a cuspidal edge if and only if p is not a ridge point of f .*
 - *The map-germ f^t at p is \mathcal{A} -equivalent to a swallowtail if and only if p is a first order ridge point of f .*
 - *The map-germ f^t at p is \mathcal{A} -equivalent to a cuspidal butterfly if and only if p is a second order ridge point of f .*
- (2) *Assume $d\kappa_+(p) = 0$. Then the following hold:*
 - *The map-germ f^t at p is \mathcal{A} -equivalent to a cuspidal lips if and only if $\text{rank}(df^t)_p = 1$ and $\det \mathcal{H}_{\kappa_+}(p) > 0$ hold.*
 - *The map-germ f^t at p is \mathcal{A} -equivalent to a cuspidal beaks if and only if p is a first order ridge point of f , $\text{rank}(df^t)_p = 1$ and $\det \mathcal{H}_{\kappa_+}(p) < 0$ hold.*

Here $\mathcal{H}_{\kappa_+}(p)$ is the Hessian matrix of κ_+ at p .

Proof. Let $f : V \rightarrow \mathbf{R}^3$ be a front, $p \in V$ a non-degenerate singular point of the second kind and ν a unit normal vector. Then we take an adapted coordinate system $(U; u, v)$ around p . By Lemma 4.1, we can take the signed area density function of parallel surface f^t with $t = 1/\kappa_+(p)$ as $\hat{\lambda}^t(u, v) = \kappa_+(u, v) - \kappa_+(p)$.

First, we prove the assertion (1). In this case, $(\hat{\lambda}^t)^{-1}(0)$ is a smooth curve near p and there exists a null vector field η^t of f^t . We set $\eta^t = \eta_1^t \partial_u + \eta_2^t \partial_v$, where η_i^t ($i = 1, 2$) are functions on U . By Lemma 2.8, $df^t(\eta^t) = \mathbf{0}$ on $S(f^t)$ is equivalent to

$$\begin{pmatrix} \hat{L} - \kappa_+(v\hat{E} - \varepsilon\hat{F}) & \hat{M} - \kappa_+\hat{F} \\ v(\hat{M} - \kappa_+\hat{F}) - \varepsilon(\hat{N} - \kappa_+\hat{G}) & \hat{N} - \kappa_+\hat{G} \end{pmatrix} \begin{pmatrix} \eta_1^t \\ \eta_2^t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

holds on $S(f^t)$. Thus the null vector field η^t can be taken as the principal vector \mathbf{v} as in (3.2) with respect to κ_+ restricted to $S(f^t)$. Under these conditions, the equation $(\eta^t)^{(k)}\hat{\lambda}^t = \mathbf{v}^{(k)}\kappa_+$ holds for some natural number k . Thus we have the assertion (1) by Fact 2.2 (1).

Next, we prove (2). In this case, $d\kappa_+$ vanishes at p . We consider the rank of df^t at p . The Jacobian matrix J_{f^t} of f^t is $J_{f^t} = (\varphi, f_v)\mathcal{M}$ at p , where

$$(4.1) \quad \mathcal{M} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - t \begin{pmatrix} \hat{E} & \hat{F} \\ \hat{F} & \hat{G} \end{pmatrix}^{-1} \begin{pmatrix} \hat{L} & \hat{M} \\ 0 & \hat{N} \end{pmatrix} = \frac{1}{\hat{N}(\hat{E}\hat{G} - \hat{F}^2)} \begin{pmatrix} -\hat{G}^2\hat{L} & \hat{G}(\hat{F}\hat{M} - \hat{G}\hat{N}) \\ \hat{F}\hat{G}\hat{L} & -\hat{F}(\hat{F}\hat{M} - \hat{G}\hat{N}) \end{pmatrix}.$$

Since $\text{rank } \mathcal{M} = 1$, it follows that $\text{rank } (J_{f^t})_p = 1$, when $t = 1/\kappa_+(p)$, and it implies that $\text{rank } (df^t)_p = 1$. Thus there exists a non-zero vector field η^t near p such that if $q \in S(f^t)$ then $df^t(\eta^t) = \mathbf{0}$ holds at q . We can take the principal vector \mathbf{v} with respect to κ_+ as η^t , then $\eta^t\eta^t\hat{\lambda}^t = \mathbf{v}^{(2)}\kappa_+$. Moreover, we see that $\hat{\lambda}_{uu}^t = (\kappa_+)_{uu}$, $\hat{\lambda}_{uv}^t = (\kappa_+)_{uv}$, $\hat{\lambda}_{vv}^t = (\kappa_+)_{vv}$. Thus we have $\det \mathcal{H}_{\hat{\lambda}^t}(p) = \det \mathcal{H}_{\kappa_+}(p)$. By using Fact 2.2 (2) and the definition of ridge points, we have the conclusion. \square

This theorem implies that the behavior of a bounded principal curvature of fronts determines the types of singularities appearing on parallel surfaces. For regular surfaces and Whitney umbrellas, similar results are obtained in [5, 6]. By (4.1) in the proof of Theorem 4.2 and [22, Theorem 1.1], we see that a parallel surface f^t does not have D_4 singularity at p .

4.2. Geometric invariants of parallel surfaces of cuspidal edges. We shall consider geometric properties of parallel surfaces of cuspidal edges. In particular, we deal with the case that parallel surfaces have cuspidal edges. For cuspidal edges, the following normal form is obtained by Martins and Saji [15].

Fact 4.3 ([15, Theorem 3.1]). *Let $f : (\mathbf{R}^2, \mathbf{0}) \rightarrow (\mathbf{R}^3, \mathbf{0})$ be a map-germ and $\mathbf{0}$ a cuspidal edge. Then there exist a diffeomorphism-germ $\theta : (\mathbf{R}^2, \mathbf{0}) \rightarrow (\mathbf{R}^2, \mathbf{0})$ and an isometry-germ $\Theta : (\mathbf{R}^3, \mathbf{0}) \rightarrow (\mathbf{R}^3, \mathbf{0})$ satisfying*

$$(4.2) \quad \Theta \circ f \circ \theta(u, v) = \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{v^2}{2}, \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3 \right) + h(u, v),$$

where $b_{20} \geq 0$, $b_{03} \neq 0$ and

$$h(u, v) = (0, u^4h_1(u), u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^4h_5(u, v)),$$

with $h_i(u)$ ($1 \leq i \leq 4$), $h_5(u, v)$ smooth functions.

Let $f : V \rightarrow \mathbf{R}^3$ be a normal form as in (4.2). Suppose that $b_{20}(= \kappa_\nu(0)) \neq 0$ and $b_{03}(= \kappa_c(0)) > 0$, that is, κ_+ is a bounded C^∞ -function near $\mathbf{0}$ and $\kappa_+(\mathbf{0}) \neq 0$.

Proposition 4.4. *Let $f : V \rightarrow \mathbf{R}^3$ be a normal form of a cuspidal edge as in (4.2) and f^t its parallel surface, where $t = 1/\kappa_+(\mathbf{0})$. We assume that f^t has a cuspidal edge at $\mathbf{0}$, that is, $4b_{12}^3 + b_{30}b_{03}^2 \neq 0$. Then it holds that the limiting normal curvature κ_ν^t and the singular curvature κ_s^t can be written as*

$$\kappa_\nu^t = -b_{20}(= -\kappa_\nu), \quad \kappa_s^t = \varepsilon \frac{b_{20}(-4b_{12}(b_{30} - a_{20}b_{12}) + a_{20}(4b_{12}^2 + a_{20}b_{03}^2))}{4b_{12}^3 + b_{30}b_{03}^2}$$

at $\mathbf{0}$, where $\varepsilon = \pm 1$ is determined as below.

Since $4b_{12}^3 + b_{30}b_{03}^2 = 0$ implies $\mathbf{0}$ is a ridge point, if $4b_{12}^3 + b_{30}b_{03}^2 = 0$ holds, then f^t has a swallowtail or worse at $\mathbf{0}$ (cf. [28]).

Proof. Let $\sigma : (-\delta, \delta) \ni s \mapsto \sigma(s) = (u(s), v(s)) \in V$ be a singular curve of the parallel surface f^t of f satisfying $\sigma(0) = \mathbf{0}$ and

$$\hat{\lambda}^t \circ \sigma(s) = \kappa_+(u(s), v(s)) - \kappa_+(\mathbf{0}) = 0,$$

where $t = 1/\kappa_+(p)$ and $\delta > 0$ is a sufficiently small. Setting $\hat{\sigma}(s) = f^t \circ \sigma(s)$, we see

$$\kappa_\nu^t(0) = \frac{\langle \hat{\sigma}''(0), \nu(\sigma(0)) \rangle}{\|\hat{\sigma}'(0)\|^2} = \frac{-\langle \hat{\sigma}'(0), \nu'(\sigma(0)) \rangle}{\|\hat{\sigma}'(0)\|^2} = \frac{-\frac{(2b_{12}u'(0) + b_{03}v'(0))^2}{4b_{20}}}{\frac{(2b_{12}u'(0) + b_{03}v'(0))^2}{4b_{20}^2}} = -b_{20},$$

since

$$\begin{aligned}\nu(\sigma(0)) &= (0, 0, 1), \quad \hat{\sigma}'(0) = \left(0, \frac{-2b_{12}u'(0) - b_{03}v'(0)}{2b_{20}}, 0\right), \\ \nu'(\sigma(0)) &= \left(-b_{20}u'(0), \frac{-2b_{12}u'(0) - b_{03}v'(0)}{2}, 0\right)\end{aligned}$$

(see Lemma 2.6). Thus we have that $\kappa_\nu^t(0) = -b_{20} = -\kappa_\nu(0)$.

Next, we consider the singular curvature κ_s^t . We now assume $(\kappa_+)_u(\mathbf{0}) \neq 0$. Then σ can be locally parametrized by the form $\sigma(v) = (u(v), v)$ around $\mathbf{0}$ by the implicit function theorem. Since $(\kappa_+)_u \neq 0$, we have $u' = -(\kappa_+)_v/(\kappa_+)_u$ and

$$u'' = \frac{-(\kappa_+)_{uu}(\kappa_+)_v^2 + 2(\kappa_+)_{uv}(\kappa_+)_u(\kappa_+)_v - (\kappa_+)_{vv}(\kappa_+)_u^2}{(\kappa_+)_u^3},$$

by differentiating $\hat{\lambda}^t \circ \sigma(v) = \kappa_+(u(v), v) - \kappa_+(\mathbf{0}) = 0$. Thus

$$\begin{aligned}\|\hat{\sigma}'\|^3 &= \left(\frac{(4b_{12}^3 + b_{30}b_{03}^2)^2}{4b_{20}^2b_{03}^2(b_{30} - a_{20}b_{12})^2}\right)^{3/2} \left(= \left|\frac{\mathbf{v}\kappa_+}{\kappa_+(\kappa_+)_u}\right|^3\right), \\ \det(\hat{\sigma}', \hat{\sigma}'', \nu) &= \frac{(4b_{12}^3 + b_{30}b_{03}^2)^2 (-4b_{12}(b_{30} - a_{20}b_{12}) + a_{20}(4b_{12}^2 + a_{20}b_{03}^2))}{8b_{20}^2b_{03}^3(b_{30} - a_{20}b_{12})^3}\end{aligned}$$

hold at $\mathbf{0}$ (see (4.3) and [28, Lemma 2.2]). We set $\varepsilon = 1$ (resp. $= -1$) if $\mathbf{v}\kappa_+$ and $(\kappa_+)_u$ have the same sign (resp. opposite sign). Thus we have

$$\kappa_s^t = \varepsilon \frac{b_{20}(-4b_{12}(b_{30} - a_{20}b_{12}) + a_{20}(4b_{12}^2 + a_{20}b_{03}^2))}{4b_{12}^3 + b_{30}b_{03}^2}$$

at $\mathbf{0}$. If $(\kappa_+)_v \neq 0$ at $\mathbf{0}$, we have the same formulation as above. \square

4.3. Constant principal curvature lines and exactly cusped points of cuspidal edges. Let $f : V \rightarrow \mathbf{R}^3$ be a front, ν a unit normal vector and p as non-degenerate singular point. Suppose that κ_+ is bounded at p and $\kappa_+(p) \neq 0$. We set $\hat{\lambda}^t(u, v) = \kappa_+(u, v) - \kappa_+(p)$. The zero-set of this function gives the singular curve of the parallel surface f^t of f , where $t = 1/\kappa_+(p)$ (Lemma 4.1). We call the curve given by $\hat{\lambda}^t = \kappa_+ - \kappa_+(p) = 0$ a *constant principal curvature (CPC) line with the value of $\kappa_+(p)$* (cf. [5, 6]). In this case, the CPC line is a regular curve since $d\hat{\lambda}^t(p) \neq 0$. In [5, 6], CPC lines for regular surfaces and Whitney umbrellas, and relations between singularities of parallel surfaces and the behavior of CPC lines are investigated. For intrinsic properties of Whitney umbrellas, see [7, 8].

Let $f : V \rightarrow \mathbf{R}^3$ be a front, $p \in V$ a cuspidal edge and assume that κ_+ is bounded near p . The condition $\eta\kappa_+ = 0$ at p implies that the CPC line is tangent to the null vector field η of f at p . Moreover, the image $f(S(f^t))$ of the set of singular points of the parallel surface f^t by f is cusped at p . We call such a point an *exactly cusped point for the constant principal curvature (CPC) line*.

Proposition 4.5. *Let $f : V \rightarrow \mathbf{R}^3$ be a front and p a cuspidal edge. Suppose that κ_+ (resp. κ_-) is bounded at p . Then $\eta\kappa_+(p) = 0$ (resp. $\eta\kappa_-(p) = 0$) implies $\kappa_s(p) \leq 0$.*

Proof. By using a normal form as in (4.2), we have

$$(4.3) \quad (\kappa_+)_u = b_{30} - a_{20}b_{12}, \quad (\kappa_+)_v = -(4b_{12}^2 + a_{20}b_{03}^2)/2b_{03}$$

at $\mathbf{0}$. Since the null vector is $\eta = \partial_v$ for a normal form (4.2), the relation $(\kappa_+)_v = \eta\kappa_+$ holds. Hence $\eta\kappa_+(\mathbf{0}) = 0$ if and only if $4b_{12}^2 + a_{20}b_{03}^2 = 0$. This implies that

$$\kappa_s(0) = a_{20} = -\frac{4b_{12}^2}{b_{03}^2} \leq 0.$$

Thus we obtain the assertion. \square

Relations between the Gaussian curvature and the singular curvature are stated in [25, Theorem 3.1].

Proposition 4.6. *Let $f : V \rightarrow \mathbf{R}^3$ be a front, p a cuspidal edge, γ a singular curve and η a null vector field. Assume that κ_+ is bounded near p , $\kappa_+(p) \neq 0$ and p is not a ridge point of f . Then the cusp-directional torsion κ_t^t of f^t vanishes at p if and only if $\eta\kappa_+$ vanishes at p , namely, p is an exactly cusped point, where $t = 1/\kappa_+(p)$.*

Proof. Let $f : V \rightarrow \mathbf{R}^3$ be a normal form as in (4.2) and σ be a singular curve of f^t satisfying $\hat{\lambda}^t(\sigma) = 0$. We assume that $(\kappa_+)_u(\mathbf{0}) \neq 0$. Then we can take $\sigma(v) = (u(v), v)$. Let $\mathbf{w} = u'\partial_u + \partial_v$ denote a vector field tangent to σ , where $u' = -(\kappa_+)_v/(\kappa_+)_u$. The pair (\mathbf{w}, \mathbf{v}) gives an adapted pair of vector fields in the sense of [15]. Moreover, $\langle \mathbf{w}f^t, \mathbf{v}f^t \rangle = 0$ holds at $\mathbf{0}$. By [15, (5.1)], we have

$$(4.4) \quad \kappa_t^t(\mathbf{0}) = \frac{\det(\mathbf{w}f^t, \mathbf{v}f^t, \mathbf{w}\mathbf{v}f^t)}{\|\mathbf{w}f^t \times \mathbf{v}f^t\|^2}(\mathbf{0}) = \frac{b_{20}^2(4b_{12}^2 + a_{20}b_{03}^2)}{4b_{12}^3 + b_{30}b_{03}^2}.$$

Comparing with (4.3) and (4.4), we obtain the result. \square

We now consider the case that $(\kappa_+)_u = 0$ at p . Since this is equivalent to $\kappa'_\nu = 0$ at p , we call such a point an *extrema of the limiting normal curvature* κ_ν . Therefore we have three special points (landmarks in the sense of [21]) on cuspidal edges which have special relations between the singular curve and the CPC line (see Figure 1). It seems that exactly cusped points have not appeared in the literature.

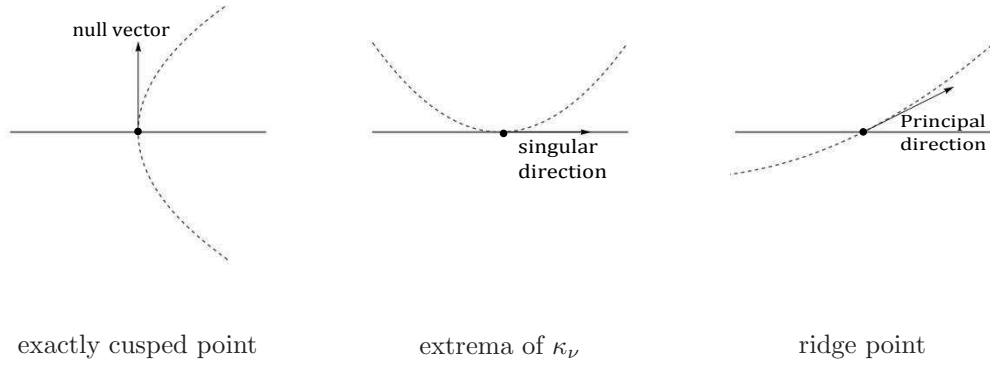


FIGURE 1. Figures of the singular curve and the CPC line near cuspidal edge. The solid curve is the singular curve and the dotted one is the CPC line through p .

5. EXTENDED DISTANCE SQUARED FUNCTIONS ON WAVE FRONTS

We consider the extended distance squared functions on fronts. We study relations between singularities of extended distance squared functions and principal curvatures. For singularities of distance squared functions on surfaces with other corank one singularities, see [6, 14].

Let $f : V \rightarrow \mathbf{R}^3$ be a front, ν a unit normal vector and p be a non-degenerate singular point of the second kind. (For cuspidal edges, see [28].) Assume that κ_+ is bounded at p and $\kappa_+(p) \neq 0$ in this section.

We set the function $\psi : V \rightarrow \mathbf{R}$ as

$$(5.1) \quad \psi(u, v) = -\frac{1}{2}(\|\mathbf{x}_0 - f(u, v)\|^2 - t_0^2),$$

where $\mathbf{x}_0 \in \mathbf{R}^3$ and $t_0 \in \mathbf{R} \setminus \{0\}$. We call ψ the *extended distance squared function with respect to \mathbf{x}_0* .

Lemma 5.1. *For the function ψ as in (5.1), $\psi(p) = \psi_u(p) = \psi_v(p) = 0$ if $\mathbf{x}_0 = f(p) + t_0\nu(p)$.*

Proof. Let us take an adapted coordinate system $(U; u, v)$ centered at p with the null vector field $\eta = \partial_u + \varepsilon(u)\partial_v$. In this case, $\psi(p) = 0$ follows from (5.1). By direct computations, we have $\psi_u = \langle \mathbf{x}_0 - f, v\varphi - \varepsilon f_v \rangle$ and $\psi_v = \langle \mathbf{x}_0 - f, f_v \rangle$. Since $\langle \varphi, \nu \rangle = \langle f_v, \nu \rangle = 0$ at p , we get the conclusion. \square

Take $\mathbf{x}_0 = f(p) + t_0\nu(p)$. We are interested in the case of $t_0 = 1/\kappa_+(p)$, because \mathbf{x}_0 corresponds to the image of a singular point of a parallel surface f^t with $t = 1/\kappa_+(p)$, that is, \mathbf{x}_0 coincides with a focal point of f at p . In such a case, ψ measures contact of f with the principal curvature sphere at p (cf. [9, 14]).

Proposition 5.2. *If $\mathbf{x}_0 = f(p) + \nu(p)/\kappa_+(p)$ and $t_0 = 1/\kappa_+(p)$, then $j^2\psi = 0$ holds, where $j^2\psi$ is the 2-jet of ψ at p .*

Proof. Take an adapted coordinate system $(U; u, v)$ around p . By Lemma 5.1, we see that $\psi = \psi_u = \psi_v = 0$ at p . By direct calculations show

$$\begin{aligned}\psi_{uu} &= -\|v\varphi - \varepsilon f_v\|^2 + \langle \mathbf{x}_0 - f, v\varphi_u - \varepsilon' f_v - \varepsilon f_{uv} \rangle, \\ \psi_{uv} &= -\langle f_v, v\varphi - \varepsilon f_v \rangle + \langle \mathbf{x}_0 - f, \varphi + v\varphi_v - \varepsilon f_{vv} \rangle, \\ \psi_{vv} &= -\|f_v\|^2 + \langle \mathbf{x}_0 - f, f_{vv} \rangle.\end{aligned}$$

Thus we have $\psi_{uu} = \psi_{uv} = 0$ at p since $\langle f_v, \nu \rangle = \langle \varphi, \nu \rangle = 0$. Moreover, it follows that $\psi_{vv} = -\hat{G} + \langle \nu, f_{vv} \rangle / \kappa_+(p) = -\hat{G} + \hat{N} / \kappa_+(p) = 0$ at p since $1/\kappa_+(p) = \hat{G}(p)/\hat{N}(p)$. Thus we have the assertion. \square

Proposition 5.2 implies that ψ may have a D_4 singularity at p if \mathbf{x}_0 coincides with the focal point of f at p , where a function-germ $h : (\mathbf{R}^2, \mathbf{0}) \rightarrow (\mathbf{R}, 0)$ has a D_4 singularity at $\mathbf{0}$ if h is \mathcal{R} -equivalent to the germ $(u, v) \mapsto u^3 \pm uv^2$ at $\mathbf{0}$ (cf. [2, pages 264 and 265]). Therefore we consider the condition that ψ has a D_4 singularity at p in terms of geometric properties of f .

For cuspidal edges, suppose that κ_+ is bounded near p and $\kappa_+(p) \neq 0$. Then the function ψ with $\mathbf{x}_0 = f(p) + \nu(p)/\kappa_+(p)$ and $t_0 = 1/\kappa_+(p)$ has a D_4 singularity if and only if $\kappa_i(p)(4\kappa_t(p)^3 + \kappa_i(p)\kappa_c(p)^2) = 0$ ([28, Theorem 3.3]). Here κ_i is a geometric invariant called the *edge inflectional curvature* defined in [15, Section 5.3]. We remark that Oset Sinha and Tari [19] study singularities of height functions and orthogonal projections of cuspidal edges.

Let $f : V \rightarrow \mathbf{R}^3$ be a front and p a singular point of the second kind. For a function $\psi : V \rightarrow \mathbf{R}$, set

$$(5.2) \quad \Delta_\psi = ((\psi_{uuu})^2(\psi_{vvv})^2 - 6\psi_{uuu}\psi_{uvv}\psi_{uvv}\psi_{vvv} - 3(\psi_{uvv})^2(\psi_{uvv})^2 + 4(\psi_{uvv})^3\psi_{vvv} + 4\psi_{uuu}(\psi_{uvv})^3)(p).$$

It is known that the function ψ is \mathcal{R} -equivalent to $u^3 + uv^2$ (resp. $u^3 - uv^2$) if and only if $j^2\psi = 0$ and $\Delta_\psi > 0$ (resp. $\Delta_\psi < 0$) hold (see [22, Lemma 3.1], see also [5, Theorem 4.2]).

Theorem 5.3. *Let $f : V \rightarrow \mathbf{R}^3$ be a front and p a singular point of the second kind. Suppose that κ_+ is bounded near p and $\kappa_+(p) \neq 0$. Then ψ as in (5.1) with $\mathbf{x}_0 = f(p) + \nu(p)/\kappa_+(p)$ and $t_0 = 1/\kappa_+(p)$ has a D_4 singularity at p if and only if p is not a ridge point of f .*

To prove this theorem, we take a special adapted coordinate system $(U; u, v)$ centered at p called a *strongly adapted coordinate system* which satisfies $\langle f_{uv}, f_v \rangle = 0$ at p (see [16, Definition 3.6]). Under this coordinate system, we see that $\hat{F} = \hat{G}_u = 0$ at p since $\varphi(p) = f_{uv}(p)$. We prepare lemma.

Lemma 5.4. *Under the above conditions, $\Delta_\psi \neq 0$ if and only if*

$$(5.3) \quad 4\psi_{uvv}\psi_{vvv} - 3(\psi_{uvv})^2 = \frac{4\hat{G}}{\hat{N}^2}(\hat{L}(\hat{G}\hat{N}_v - \hat{G}_v\hat{N}) - \hat{G}\hat{M}(\hat{N}_u + \hat{M})) \neq 0$$

at p .

Proof. We take a strongly adapted coordinate system $(U; u, v)$ around p . Direct calculations show that

$$\psi_{uuu} = t_0\langle \nu, f_{uuu} \rangle, \quad \psi_{uvv} = t_0\langle \nu, f_{uvv} \rangle - \langle f_v, f_{uu} \rangle$$

hold at p , where $t_0 = 1/\kappa_+(p) = \hat{G}(p)/\hat{N}(p)$. Since $f_{uu} = -\varepsilon' f_v$, $f_{uuu} = -\varepsilon'' f_v - 2\varepsilon' \varphi$ and $f_{uvv} = \varphi_u - \varepsilon' f_{vv}$ at p , it follows that $\psi_{uuu} = 0$ and

$$(5.4) \quad \psi_{uvv} = t_0\langle \nu, \varphi_u \rangle + \varepsilon'(-t_0\langle \nu, f_{vv} \rangle + \|f_v\|^2) = t_0\langle \nu, \varphi_u \rangle = \hat{G}\hat{L}/\hat{N} \neq 0$$

hold at p . Thus Δ_ψ as in (5.2) can be written as

$$\Delta_\psi = (\psi_{uvv}(p))^2(4\psi_{uvv}(p)\psi_{vvv}(p) - 3(\psi_{uvv}(p))^2).$$

This implies that $\Delta_\psi \neq 0$ if and only if $4\psi_{uvv}(p)\psi_{vvv}(p) - 3(\psi_{uvv}(p))^2 \neq 0$.

We consider the form of $4\psi_{uvv}(p)\psi_{vvv}(p) - 3(\psi_{uvv}(p))^2 \neq 0$. By direct computations, we have

$$\psi_{uvv} = t_0 \langle \nu, f_{uvv} \rangle, \quad \psi_{vvv} = t_0 \langle \nu, f_{vvv} \rangle - 3 \langle f_v, f_{vv} \rangle$$

at p . Since $f_{uvv} = 2\varphi_v$ at p , it follows that

$$(5.5) \quad \psi_{uvv}(p) = 2t_0 \hat{M}(p) = \frac{2\hat{G}(p)\hat{M}(p)}{\hat{N}(p)}.$$

We now deal with $\psi_{vvv}(p)$. It follows that $\langle \nu, f_v \rangle = 0$ and $\langle \nu, f_{vv} \rangle = -\langle \nu_v, f_v \rangle = \hat{N}$ on U . So $\langle \nu, f_{vvv} \rangle = \hat{N}_v - \langle \nu_v, f_{vv} \rangle$ holds. By Lemma 2.8, $\langle \nu_v, f_{vv} \rangle$ is written as

$$\langle \nu_v, f_{vv} \rangle = -\frac{\hat{M}}{\hat{E}} \langle \varphi, f_{vv} \rangle - \frac{\hat{N}}{\hat{G}} \langle f_v, f_{vv} \rangle$$

at p . On the other hand, $\hat{N}_u = \langle \nu_u, f_{vv} \rangle + \langle \nu, f_{uvv} \rangle = -\hat{L} \langle \varphi, f_{vv} \rangle / \hat{E} + 2\hat{M}$ at p by Lemma 2.8. Hence we have $\langle \varphi, f_{vv} \rangle = -\hat{E}(\hat{N}_u - 2\hat{M}) / \hat{L}$ and

$$(5.6) \quad \psi_{vvv} = \frac{\hat{G}\hat{N}_v - \hat{G}_v\hat{N}}{\hat{N}} - \frac{\hat{G}\hat{M}(\hat{N}_u - 2\hat{M})}{\hat{L}\hat{N}}$$

at p , where we used $2\langle f_v, f_{vv} \rangle = \hat{G}_v$. By (5.4), (5.5) and (5.6), $4\psi_{uvv}\psi_{vvv} - 3(\psi_{uvv})^2$ can be written as

$$4\psi_{uvv}\psi_{vvv} - 3(\psi_{uvv})^2 = \frac{4\hat{G}}{\hat{N}^2} (\hat{L}(\hat{G}\hat{N}_v - \hat{G}_v\hat{N}) - \hat{G}\hat{M}(\hat{N}_u + \hat{M}))$$

at p . Thus we have the assertion. \square

Proof of Theorem 5.3. Let us take a strongly adapted coordinate system $(U; u, v)$ centered at p . Then we note that $\hat{F} = \hat{G}_u = 0$ holds at p . The differentials $(\kappa_+)_u$ and $(\kappa_+)_v$ are given by

$$(\kappa_+)_u = \frac{\hat{N}_u}{\hat{G}}, \quad (\kappa_+)_v = \frac{-\hat{G}\hat{M}^2 + \hat{L}(\hat{G}\hat{N}_v - \hat{G}_v\hat{N})}{\hat{G}^2\hat{L}}$$

at p . Since the principal vector \mathbf{v} as in (3.2) is written as $\mathbf{v} = (-\hat{M}, \hat{L})$ at p , we have

$$(5.7) \quad \begin{aligned} \mathbf{v}\kappa_+(p) &= -\hat{M}(p)(\kappa_+)_u(p) + \hat{L}(p)(\kappa_+)_v(p) \\ &= \frac{1}{\hat{G}(p)^2} (\hat{L}(p)(\hat{G}(p)\hat{N}_v(p) - \hat{G}_v(p)\hat{N}(p)) - \hat{G}(p)\hat{M}(p)(\hat{N}_u(p) + \hat{M}(p))). \end{aligned}$$

Comparing (5.7) and (5.3) in Lemma 5.4, $\mathbf{v}\kappa_+(p) \neq 0$, namely, p is not a ridge point of f if and only if $4\psi_{uvv}(p)\psi_{vvv}(p) - 3(\psi_{uvv}(p))^2 \neq 0$. This implies that the number Δ_ψ defined as (5.2) does not vanish by Lemma 5.4. \square

We remark that the condition that f is a front in Theorem 5.3 is needed for ψ to have a D_4 singularity at p . In fact, for a frontal $f : V \rightarrow \mathbf{R}^3$ with a singular point of the admissible second kind p , we have the following.

Proposition 5.5. *Let $f : V \rightarrow \mathbf{R}^3$ be a frontal but not a front and p a singular point of the admissible second kind. Then ψ with $\mathbf{x}_0 = f(p) + \nu(p)/\kappa_\nu(p)$ and $t_0 = 1/\kappa_\nu(p)$ does not have a D_4 singularity at p .*

Proof. Let us take an adapted coordinate system $(U; u, v)$ centered at p with the null vector field $\eta = \partial_u + \varepsilon(u)\partial_v$. By the proof of Lemma 5.4, Δ_ψ vanishes automatically if f is not a front at p . \square

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